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From continuous Painlevé IV to the asymmetric discrete Painlevé I

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Abstract. We examine the discrete equation that is obtained from the Schlesinger transformations of the continuous Painlevé IV equation. The use of the Schlesinger transformations naturally establishes the Lax pair of the discrete equation which is an asymmetric form of d-P_I and has P_{II} as continuous limit. We analyse the transformations of the discrete equation and show that they lead to a self-dual description where the discrete equation and the Schlesinger's play the same role. Finally, using the relation between P_{IV} and asymmetric d-P_I we examine in detail the special solutions of both equations.

1. Introduction

The derivation of the general form of the discrete Painlevé equations ($\mathbb{P}$) based on the singularity confinement [1] discrete integrability criterion led to an intriguing feature. For most of the discrete $\mathbb{P}$'s the general form contains periodic terms [2]. The simplest case is that of binary (even–odd) dependence, manifesting itself through the presence of terms of the form $(-1)^k$ in the equation. Thus the full form of d-P_I is

$$X_{k-1} + X_k + X_{k+1} = t + \frac{k\alpha + \beta + \gamma(-1)^k}{X_k}. \tag{1.1}$$

However, cases of higher (i.e. ternary, quaternary, etc) dependence are known to exist. The standard attitude when these terms were discovered was to simply ignore them. The naive argument was that '$(-1)^k$ does not possess a continuous limit'. For equation (1.1), we obtained the continuous limit to P_I by putting $\gamma = 0$ [2].

However, this approach is clearly too crude. The correct interpretation is to consider that a different constant enters the equation for the even- and odd-numbered x's and thus write the mapping as a system where the even and odd terms are separated. We now define $x_n = X_{2n}$, $y_n = X_{2n+1}$ and introduce $z_n = 2n\alpha + z_0$, with $z_0 = \beta + \alpha/2$ and $c = \gamma - \alpha/2$. We thus obtain:

$$y_{n-1} + x_n + y_n = t + \frac{z_n + c}{x_n} \tag{1.2a}$$

$$x_n + y_n + x_{n+1} = t + \frac{z_n - c}{y_n}. \tag{1.2b}$$

We call equation (1.2) the 'asymmetric' d-P_I. (The terminology perhaps needs some explanation. At the autonomous limit $\alpha \to 0$ the mappings (1.1) and (1.2) transfer to Quispel–Roberts–Thompson (QRT) forms [3] which were shown to be solvable in terms of elliptic functions. Expressions such as (1.1) with $\gamma = 0$ belong to the 'symmetric' QRT family, while those such as (1.2) belong to the 'asymmetric' family.) However, the mapping (1.2) is not just another form of d-P_I. This fact was realized for the first time in the case of the 'asymmetric' d-P_{II}. In [4], we have shown that the latter system is in fact a discrete form of the Painlevé III equation and so, it should have been called d-P_{III}. This does not mean that it is equivalent to the other already known discrete form of P_{III} [2] which is in fact a q-P_{III}; the two equations only have the same continuous limit. (The q-P_{III} equation was also known to possess an 'asymmetric' form which was recently shown to be a discrete form of P_{VI} [5].)

Before proceeding further let us spend a few lines to make the distinction clear between the two types of equations we hinted at above. Equations such as (1.1) are *difference* equations. The independent variable enters in an additive way: $z_n = n\delta + z_0$, and thus the mapping is a recurrence that relates the value of the function at points z and $z \pm \delta$. However, a second type of equation exists. The q-P_{III} we mentioned above is one such example, but there are many more. For instance several q-P_I's are known [6]:

$$x_n^\sigma x_{n-1} x_{n+1} = 1 + \beta q^n x_n \tag{1.3}$$

(for $\sigma = 0, 1, 2$). Here the independent variable enters in a multiplicative way $z = q^n z_0$ and thus the mapping relates the values of the function at points z and $zq^{\pm 1}$. Equations of this kind are called q-equations. This is sometimes emphasized through the use of the prefix q- in the name of the equation. As a matter of fact, this is at best a half-measure. It would have been preferable to use the symbol d-P for all the discrete $\mathbb{P}$'s and then distinguish between difference- and q-equations (whenever this distinction is crucial) by using the more appropriate notations δ-P's and q-P's.

Returning to the 'asymmetric' d-P_I it is clear that the equation is interesting and intriguing. We have two different methods for its derivation (singularity confinement [1] and through the Schlesinger transforms of the continuous P_{IV} [7]) and both lead to exactly the same equation. Its integrability is established beyond any doubt: its Lax pair has been obtained in [8] (although the discrete equation associated with it was not identified at the time). However, when we examine (1.2) more closely, some questions appear unavoidable. This equation possesses one genuine parameter. Thus we expect it to be a form of discrete P_{II} (a fact that will be confirmed by our analysis in what follows). If this is the case then 'asymmetric' d-P_I must possess special solutions and auto-Bäcklund transformations which *must disappear* when we consider the symmetric limit. Indeed, the latter corresponds to d-P_I which can have neither auto-Bäcklund transformations nor special solutions.

2. Derivation of the asymmetric d-P_I: Lax pair and auto-Bäcklund transformations

In this section, we shall present the derivation of the asymmetric d-P_I starting from the continuous P_{IV} equation. We shall take advantage of this derivation in order to present the deep relation that exists between continuous and discrete (*difference*) equations. Let us start with the Lax pair of a continuous $\mathbb{P}$. It has the general form

$$\psi_\zeta = A\psi \tag{2.1a}$$

$$\psi_t = B\psi \tag{2.1b}$$

where ζ is the spectral parameter and A, B are matrices depending explicitly on ζ and the dependent as well as the independent variables w and t. The continuous $\mathbb{P}$ equation is obtained from the compatibility condition $\psi_{\zeta t} = \psi_{t\zeta}$ leading to

$$A_t - B_\zeta + AB - BA = 0. \tag{2.2}$$

In general, the $\mathbb{P}$ equation depends on parameters $(\alpha, \beta, \ldots)$ which are associated with the monodromy exponents θ_i appearing explicitly in the Lax pair. The Schlesinger transform relates two solutions Ψ and Ψ' of the isomonodromy problem for the equation at hand corresponding to different sets of parameters $(\alpha, \beta, \ldots)$ and $(\alpha', \beta', \ldots)$. The main characteristic of these transforms is that the monodromy exponents (at the singularities of the associated linear problem), related to the sets $(\alpha, \beta, \ldots)$ and $(\alpha', \beta', \ldots)$ differ by integers (or half-integers). The general form of a Schlesinger transformation is

$$\psi' = R\psi \tag{2.3}$$

where R is again a matrix depending on ζ, w, t and the monodromy exponents θ_i.

The important remark is that (2.1a) together with (2.3) constitute *the Lax pair of a discrete equation*. The latter is obtained from the compatibility conditions,

$$R_\zeta + RA - A'R = 0. \tag{2.4}$$

Thus the difference equations are intimately related to the continuous ones. We believe that this result is of very wide applicability and, although this has not been done yet, that we can obtain a classification of all difference discrete equations through their relations to the continuous ones. This classification would have another beneficial consequence: it would put an end to the proliferation of the discrete $\mathbb{P}$'s, since there are only so many Schlesinger's of continuous $\mathbb{P}$'s. Still, we must bear in mind that it may well happen that some higher Garnier problems may lead, through their Schlesinger's, to new simple difference $\mathbb{P}$'s. Thus the relation between difference and continuous $\mathbb{P}$'s must be understood in a broader sense.

Another remark is in order at this point. While the case of difference discrete equations seems settled, this does not encompass all discrete equations. (After all, it would have been disappointing if all difference equations were just byproducts of continuous ones.) There remain the q-discrete $\mathbb{P}$'s; these cannot be obtained from the continuous ones, but only from higher q-$\mathbb{P}$'s. Just as the Schlesinger's of difference discrete $\mathbb{P}$'s can generate equations of the same type, the same holds true for the q-$\mathbb{P}$'s [9]. What makes matters more complicated is that in contrast to the continuous case, the general q-Garnier problem has not yet been fully derived. Independently of this technical point, the argument above reinforces the fundamental nature of the discrete equations in particular of the q-form.

After these considerations of general nature let us turn back to the $\mathrm{P_{IV}}$ equation. Its Lax pair is well known [10]:

$$A = \zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} t & u \\ \frac{2}{u}(v - \theta_0 - \theta_\infty) & -t \end{pmatrix} + \zeta^{-1} \begin{pmatrix} \theta_0 - v & -\frac{uw}{2} \\ \frac{2v}{uw}(v - 2\theta_0) & -(\theta_0 - v) \end{pmatrix} \tag{2.5a}$$

$$B = \zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & u \\ \frac{2}{u}(v - \theta_0 - \theta_\infty) & 0 \end{pmatrix}. \tag{2.5b}$$

The compatibility leads to

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= -4v + w^2 + 2tw + 4\theta_0 \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= -u(w + 2t) \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{2v^2}{w} + \left(\frac{4\theta_0}{w} - w\right) v + (\theta_0 + \theta_\infty) w \end{aligned} \tag{2.6}$$

which results in $\mathrm{P_{IV}}$:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 + \frac{3}{2}w^3 + 4tw^2 + 2(t^2 + a)w + \frac{b}{w}. \tag{2.7}$$

The parameters a, b are related to the monodromy exponents θ_0, θ_∞ through

$$a = 1 - 2\theta_\infty \qquad b = -8\theta_0^2. \tag{2.8}$$

The Schlesinger transforms of P_{IV} describe the evolution of the two monodromy exponents θ_0, θ_∞. There are several transformations one can construct and the result will, of course, depend on the one we choose. In [7], we have chosen the two Schlesinger's below, which turned out to be adequate for the problem at hand. We have

$$R_1 = \zeta^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \zeta^{-1/2}\begin{pmatrix} \frac{v}{w} & \frac{u}{2} \\ \frac{2v}{uw} & 1 \end{pmatrix} \tag{2.9}$$

corresponding to the evolution $\underline{\theta}_\infty = \theta_\infty - \frac{1}{2}$, $\underline{\theta}_0 = \theta_0 + \frac{1}{2}$, and

$$R_2 = \zeta^{1/2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \zeta^{-1/2}\begin{pmatrix} 1 & \frac{uw}{2v} \\ -\frac{v-\theta_0-\theta_\infty}{u} & -\frac{w(v-\theta_0-\theta_\infty)}{2v} \end{pmatrix} \tag{2.10}$$

associated with $\tilde{\theta}_0 = \theta_0 + \frac{1}{2}$, $\tilde{\theta}_\infty = \theta_\infty + \frac{1}{2}$. Each of the R_1, R_2 leads to a discrete equation which turns out to be the asymmetric d-P_I. Let us start from R_1 and apply the compatibility (2.4). The result is

$$\underline{w} = \left(\frac{2v}{w}\left(t - \frac{v}{w}\right) + v + \theta_0 - \theta_\infty + 1\right)\left(\frac{v}{w} - \frac{w}{2} - t\right)^{-1} \tag{2.11a}$$

$$\underline{v} = -\frac{2v}{w}\left(t - \frac{v}{w}\right) - v + \theta_0 + \theta_\infty. \tag{2.11b}$$

We put $w = -2x_n$, $v = 2(\theta_0 - y_n x_n)$ (and $\underline{w} = -2x_{n-1}$, $\underline{v} = 2(\theta_0 + \frac{1}{2} - y_{n-1}x_{n-1})$) and find

$$y_n + y_{n+1} = t - x_n + \frac{\theta_0}{x_n} \tag{2.12a}$$

$$x_n + x_{n-1} = t - y_n + \frac{\theta_0 - \theta_\infty}{2y_n} \tag{2.12b}$$

where we recognize the asymmetric d-P_I equation (1.2) (since both θ_∞ and $-\theta_0$ grow linearly in n with the same coefficient under the action of R_1).

For R_2 we have

$$\tilde{w} = -2t + \frac{2v}{w} - w + \frac{w}{v}(\theta_0 + \theta_\infty) \tag{2.13a}$$

$$\tilde{v} = -v + tw + 3\theta_0 + \theta_\infty + 1 + \frac{w^2(\theta_0+\theta_\infty)^2}{2v^2} + \frac{w^2}{2} - \frac{w(\theta_0+\theta_\infty)(t+w)}{v}. \tag{2.13b}$$

We must stress here that R_2 induces an evolution in a direction different from that of R_1; this is why we have used the 'bar' and 'tilde' symbols. We introduce the new variables $w = -2x_m$ and $v = -2r_m x_m$ (where x has the same relation to w as before but evolves in a different direction, and r is a new variable) and obtain

$$r_m + r_{m-1} = t - x_m - \frac{\theta_0}{x_m} \tag{2.14a}$$

$$x_m + x_{m+1} = t - r_m - \frac{\theta_0 + \theta_\infty}{2r_m} \tag{2.14b}$$

which is again the asymmetric d-P_I. Here θ_0 and θ_∞ grow linearly in m with the same coefficient.

The relation of the variables of asymmetric d-P_I to those of P_{IV} allows an easy derivation of the auto-Bäcklund transformation of the discrete equation. Since $x = -w/2$ in both cases, a point n_0 exists in the evolution (2.12) and m_0 in the evolution (2.14) where the corresponding w's coincide. At this point we have for the first equation,

$$xy = -\frac{v}{2} + \theta_0 \tag{2.15}$$

and for the second one,

$$xr = -\frac{v}{2}. \tag{2.16}$$

Eliminating the variable v of the continuous equation we obtain

$$xy = xr + \theta_0. \tag{2.17}$$

This is an auto-Bäcklund of the asymmetric d-$\mathrm{P_I}$ equation since it relates the couple (x, y) to the couple (x, r). In the same spirit one can derive all of the auto-Bäcklund transformations that we shall use in the next section.

Before proceeding further let us show that the asymmetric d-$\mathrm{P_I}$ is, in fact, a discrete form of d-$\mathrm{P_{II}}$. The continuous limit of (1.2) can be obtained through $x = 1 + \epsilon w + \epsilon^2 u$, $y = 1 - \epsilon w + \epsilon^2 u$, $t = 2$, $c = \epsilon^3 \mu/4$ and $z_n = 1 + \epsilon^3 n$ leading at $\epsilon \to 0$ to

$$\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} = 2w^3 + sw - (\mu + \tfrac{1}{2}) \tag{2.18}$$

where the continuous variable is given by $s = \epsilon n$ and $u = (w^2 + w' + s/2)/4$.

3. A self-dual description of the asymmetric d-$\mathrm{P_I}$ and its Schlesinger's

As we have already shown in [9], the discrete $\mathbb{P}$'s are characterized by the property of self-duality. This means that the same equation governs the evolution both along the discrete independent variable and along changes of the parameters of the equation induced by Schlesinger transformations. We have already pointed out that this is possible only when we consider the full freedom of the equation without any *ad hoc* restrictions. In what follows, we shall present the self-dual description of the asymmetric d-$\mathrm{P_I}$ which as we shall see has a very particular geometry.

We start from system (1.2) which we rewrite here:

$$y_{n-1} + x_n + y_n = t + \frac{z_n + c}{x_n} \tag{3.1a}$$

$$x_n + y_n + x_{n+1} = t + \frac{z_n - c}{y_n}. \tag{3.1b}$$

In order to simplify the presentation (and in view of the geometry to be introduced below) we shall use the shorthand notation $y_n = y$, $y_{n+1} = \bar{y}$, $y_{n-1} = \underline{y}$ and similarly for x. System (3.1) defines the evolution of the variables x, y in the n direction, along the independent discrete variable.

The dual equation to (3.1) would be the one concerning the evolution along the c-direction. It turns out that the geometry in the case of the asymmetric d-$\mathrm{P_I}$ is more subtle. First we must introduce an auxiliary (dependent) variable r, which will be related to x and y. Next, instead of considering two orthogonal directions of evolution along n and c, we consider three directions but still in a two-dimensional plane. One is the direction of n, but the other two are at $\pm 2\pi/3$ and correspond thus to a mixture of n and c. Following our shorthand notations, we will denote evolution along the $2\pi/3$ line by $\hat{x}$, $\hat{y}$ (and $\underset{\wedge}{x}$, $\underset{\wedge}{y}$ in the backward direction), while for the evolution along the $-2\pi/3$ line we shall use the notation $\tilde{x}$, $\tilde{y}$, and $\underset{\sim}{x}$, $\underset{\sim}{y}$. The precise geometry is given in figure 1. Note that the three directions

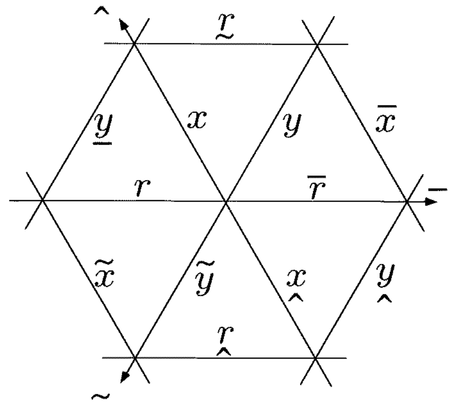

Figure 1. The geometry of the x, y, r plane.

being linearly dependent one has, for instance $\hat{\hat{x}} = \underset{\sim}{x}$. The variables x, y 'live' on a line parallel to the n-axis corresponding to $c + \alpha/2$ rather than c. In fact, their coordinates are $(z - \alpha/2, c + \alpha/2)$ for x and $(z + \alpha/2, c + \alpha/2)$ for y. Note that in (3.1) $z + c$ and $z - c$ are respectively the sums and differences of the coordinates of their respective denominators x and y.

The auxiliary function r, on the other hand, lies on a parallel to the n-axis labelled by c exactly and can only define an evolution along the two oblique axes. We shall not present all the details here concerning the derivation of r but will give the final result. We have

$$x r = \underset{\wedge}{x} \bar{r} = x y - (z + c) \tag{3.2}$$

$$r \tilde{y} = y \bar{r} = x r + 2c = x y - (z - c). \tag{3.3}$$

How can one construct the Schlesinger transformations using these expressions? It is clear from figure 1 that having x, y, we can compute r and from (x, r) we can obtain $\tilde{y}$ which lies on a line parallel to the n-axis labelled by $c - \alpha/2$. We have

$$\tilde{y} = x \frac{x y - (z - c)}{x y - (z + c)} \tag{3.4}$$

$$\underset{\wedge}{x} = y \frac{x y - (z + c)}{x y - (z - c)} \tag{3.5}$$

and, of course, $\underset{\wedge}{x} \tilde{y} = x y$. The new equations (where we have used $\tilde{\tilde{y}} = \underset{\wedge}{y}$) now read

$$\tilde{y} + \underset{\wedge}{x} + \underset{\wedge}{y} = t + \frac{z + c}{\underset{\wedge}{x}} \tag{3.6a}$$

$$\tilde{x} + \tilde{y} + \underset{\wedge}{x} = t + \frac{z - c}{\tilde{y}}. \tag{3.6b}$$

Comparing (3.6) with (3.1) it is now straightforward to check that the value of c has been shifted by $-\alpha$ from $c + \alpha/2$ to $c - \alpha/2$. Indeed in (3.6) $z + c$ and $z - c$ are respectively the sums and differences of the coordinates of the respective denominators $\underset{\wedge}{x}$ and $\tilde{y}$, which shows that the value of the second coordinate is now $c - \alpha/2$.

In order to investigate self-duality we consider the equation relating x and r along an oblique line. We find

$$\underset{\sim}{r} + x + r = t - \frac{z + c}{x} \tag{3.7a}$$

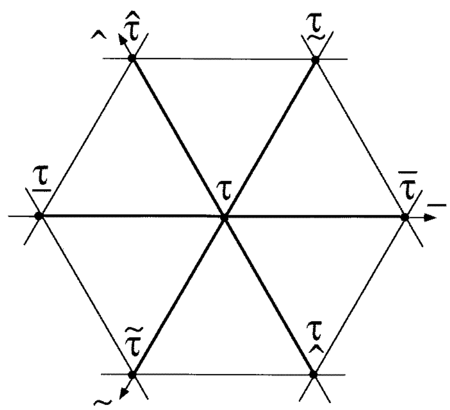

Figure 2. The τ-function plane.

$$x + r + \tilde{x} = t - \frac{2c}{r}. \tag{3.7b}$$

Owing to the particular geometry of the asymmetric d-P_{II}, self-duality cannot quite be assessed at first glance from (3.3)–(3.7). Still, it is present as we shall explain now. Let us define a coordinate system associated with the n and c evolutions. The units on each axis are such that the abscissa is $2z/\sqrt{3}$ and the ordinate $2c$. If we start with (3.7b) we remark that the numerator (over r) is $-2c$. This is precisely the opposite of the ordinate of the horizontal line that intersects the (oblique) evolution line at r. Similarly, the numerator $-(z+c)$ over x in (3.7a) is just, in the appropriate axis system rotated by $2\pi/3$, the ordinate (not its opposite) of the other oblique line that intersects the evolution line at x. Self-duality is now clear. Equation (3.3) corresponds to a rotation by $2\pi/3$ of the direction of evolution. The ordinates of the intersecting lines at x and y are respectively $-(z+c)$ and $(z-c)$. The reason why the numerator over x in (3.3a) is $(z+c)$ rather than its opposite is related to the angle $2\pi/3$ rather than $-2\pi/3$ between the evolution line and the line intersecting it, just as we already encountered in (3.7b).

We now introduce the τ-functions through

$$x = \frac{\underline{\tau}\,\utilde{\tau}}{\tau\hat{\tau}} \qquad y = \frac{\bar{\tau}\hat{\tau}}{\tau\,\utilde{\tau}} \qquad \text{and} \qquad r = \frac{\hat{\tau}\tilde{\tau}}{\tau\,\underline{\tau}}. \tag{3.8}$$

Superposing figures 1 and 2 one can see that each nonlinear variable x, y, r lies at the centre of a diamond shape, the two nearest-neighbouring τ's are those appearing in the denominator, while the two next-nearest ones appear at the numerator.

Next, we express the Miura (3.2) and (3.3) in bilinear form and find

$$\tilde{\tau}\,\utilde{\tau} - \bar{\tau}\,\underline{\tau} = -(z+c)\tau^2 \tag{3.9a}$$

$$\bar{\tau}\,\underline{\tau} - \hat{\tau}\,\underaccent{\hat}{\tau} = (z-c)\tau^2 \tag{3.9b}$$

$$\hat{\tau}\,\underaccent{\hat}{\tau} - \tilde{\tau}\,\utilde{\tau} = 2c\tau^2. \tag{3.9c}$$

As expected the Hirota–Miwa [11] (discrete Toda) equation is the bilinear Schlesinger. Self-duality can be readily assessed in (3.9). If we start with (3.9c) we remark that the coefficient of τ^2 is the ordinate $2c$ conjugate to the abscissa along the direction, (namely the horizontal one, labelled by the bar) that does not appear in the left-hand side. Equations (3.9a, b) correspond to rotations of the direction of evolution by $\pm 2\pi/3$.

4. The special solutions of asymmetric d-$\mathrm{P_I}$ and their relations to those of $\mathrm{P_{IV}}$

The construction of the special solutions of the asymmetric d-$\mathrm{P_I}$ equation is based for the major part on the special solutions of the continuous $\mathrm{P_{IV}}$. Let us start with the case where asymmetric d-$\mathrm{P_I}$ is linearizable. (As we have already shown in previous works, all d-P's, except for the various symmetric d-$\mathrm{P_I}$'s, do possess special solutions of this type.) The linearizable solutions are obtained for particular values of the parameters of the discrete equation at hand.

The general method for the construction of these special solutions has been explained in detail in [12]. Here we shall obtain these solutions by using a simple trick. We start from (1.2*b*) and assume that $1/y_n$ appearing in the right-hand side is proportional to x_n (in which case (1.2*b*) and also (1.2*a*) become linear). Comparing (1.2*a*) with (1.2*b*) we find that these two equations are indeed compatible provided $c = 0$. We then find equation $y_{n-1} + x_n = t$ and its upshift. Putting $x_n = A_n/A_{n-1}$, we can linearize the latter to

$$A_{n+1} - tA_n + z_n A_{n-1} = 0. \tag{4.1}$$

This is a discrete form of the Airy equation (already encountered in [13]). However, since x is, up to a numerical factor, the solution of $\mathrm{P_{IV}}$, this equation must also characterize the special solutions of the latter. In order to make the comparison more transparent, we will assume here that $\alpha = \frac{1}{2}$, which can be ensured with the appropriate scaling of equation (1.2). Then (4.1) is precisely *the recursion formula of the Hermite functions* and thus A_n is nothing but the Hermite functions H_{z_n}.

The functions H_z (of index $z = n + z_0$, with z_0 in general *not* an integer) are known to be related to the special solutions of $\mathrm{P_{IV}}$ [14]. Let us give here another property of the Hermite functions that will be useful in what follows. We have the differential relation,

$$\frac{\mathrm{d}H_z}{\mathrm{d}t} = zH_{z-1}. \tag{4.2}$$

As far as the discrete equation is concerned, A is defined for a *given* value of t, which is just a parameter. However, it turns out that we can consistently choose its t dependence to be exactly the one given by (4.2) in which case the correspondence between discrete and continuous is perfect. This can be done by taking the t derivative of (4.1), using (4.2) and its upshift to write everything in terms of A_n leading to a differential equation which is just the Hermite equation.

Higher special solutions do also exist for different parameter values. They can be expressed, in general, as ratios of Casorati determinants (involving the same Hermite/discrete-Airy functions) [15]. The key element here is the τ-function appearing in the bilinear formalism. The construction will be easily understood in the self-dual setting we introduced in the previous section. We start by assuming that the τ's vanish on the lower half (z, c)-plane, and that there exists a first nonvanishing line of τ's compatible with equation (3.9*c*) which means that c must be zero on this line, given that the line below this line consists only of vanishing τ's. Let us assume that the τ-functions on the $c = 0$ line have the values ϕ_n. From (3.9a, b) we have

$$\phi_{n+1}\phi_{n-1} = z_n\phi_n^2 \tag{4.3}$$

(since $c = 0$). Moreover, we can freely choose two of the ϕ's to be equal to unity, say ϕ_{n-1} and ϕ_n. In this case we have $\phi_{n+1} = z_n$, $\phi_{n-2} = z_{n-1}$, $\phi_{n+2} = z_n^2 z_{n+1}$, $\phi_{n-3} = z_{n-1}^2 z_{n-2}\ldots$ and so on. Next, we assume that $\underset{\sim}{\tau}_n = \phi_n A_n$, which, using (3.16) leads to the already obtained results $x_n = A_n/A_{n-1}$ and $y_n = z_n/x_n = z_n A_{n-1}/A_n$. In order to obtain the 'higher' solutions it is more convenient to return to a notation where the shifts become explicit. Starting from τ_n at z_n and $c=0$, we have $\hat{\tau}_n = \tau(z_n - \frac{1}{2}, \frac{1}{2})$, $\underset{\sim}{\tau}_n = \tau(z_n + \frac{1}{2}, \frac{1}{2})$. Similarly τ_n, k-times shifted in the hat-direction, is $\tau(z_n - k/2, k/2)$, while if k-times shifted in the down tilde-direction we obtain $\tau(z_n + k/2, k/2)$. (Remember $\alpha = \frac{1}{2}$ and thus $z_{n+1} - z_n = 1$.) The latter has the same prefactor $\phi(z)$ as $\tau(z, 0)$. Up to this prefactor, this τ-function can be written as a Casorati determinant as follows

$$\tau(z_n + k/2, k/2) = \phi(z_n) \begin{vmatrix} A_n & A_{n+1} & \dots & A_{n+k-1} \\ A_{n+1} & \ddots & & \vdots \\ \vdots & & & \\ A_{n+k-1} & \dots & & A_{n+2k-2} \end{vmatrix}. \tag{4.4}$$

(Note that $\tau(z_n - k/2, k/2)$ has A_{n-1} on the antidiagonal, and indeed $\hat{\tau}_n = \underset{\sim}{\tau}_{n-1} = \phi_{n-1} A_{n-1}$.) What is interesting is that once the τ-functions are expressed in terms of Casorati determinants we can construct the solutions of both asymmetric d-P_{I} and P_{IV} since the variable x is (essentially) the same for both equations.

Up to now, we have worked with a generic, noninteger z_n, which means that the Hermite functions, introduced above, do not degenerate. However, P_{IV}/asymmetric d-P_{I} possesses two different families of interesting solutions in the particular case of *integer* z_0. First, let us assume that for some value of n we have $z_n = 1$, which means that $z_{n-1} = 0$. In this case, all of the ϕ's at the left-hand side of the tilde-axis vanish. This means that the only nonvanishing τ's are those lying in a $\pi/3$ sector delimited by the bar- and the negative tilde-axes. In order to obtain the τ-functions we start by computing the A's which lie on the first line parallel to the bar-axis. We choose $A_{n-1} = 1$ and $A_n = t$ (so as to be compatible with (4.2) since $z_n = 1$). This leads to $A_{n+1} = t^2 - 1$, $A_{n+2} = t^3 - 3t$, i.e. the Hermite polynomials. (Note that the A's with indices inferior to $n-1$ cannot be determined, but since the corresponding ϕ's vanish, this indeterminacy is of no consequence.) Using the A's as basic elements we can compute the τ's through the Casorati (4.4). The latter, we stress once more, provide the rational solutions to both P_{IV} [16] and asymmetric d-P_{I}.

The second type of special solutions is obtained with a similar assumption. This time we assume that $z_n = 0$ which means that all the ϕ's to the right-hand side of the tilde-axis vanish. Thus the nonvanishing τ's live in a $2\pi/3$ sector between the negative bar- and the negative tilde-axes. For the last nonvanishing τ we can choose the normalization $\tau_n = 1$ (and thus $A_n = 1$). For the first *vanishing* τ, namely τ_{n+1}, we have of course $\tau_{n+1} = 0$, because $\phi_{n+1} = 0$, but A_{n+1} does not vanish. Using relation (4.1) we find readily that in fact $A_{n+1} = t$ since $z_n = 0$, and this is indeed compatible with (4.2). We can downshift both (4.1) and (4.2) and obtain a first-order differential equation for A_{n-1}. We find

$$A_{n-1} = -\mathrm{e}^{t^2/2} E \qquad \text{with } \frac{\mathrm{d}E}{\mathrm{d}t} = \mathrm{e}^{-t^2/2} \tag{4.5}$$

i.e. E is an error-function [17]. Once A_{n-1} is obtained, we can compute all of the remaining A's by simple differentiation through (4.2) and thus this solution of P_{IV}/asymmetric d-P_{I} involves nothing but error-functions and exponentials. Again, as in the case of the Hermite polynomials, the construction of the 'higher error-functions' solutions is straightforward if one uses the Casorati (4.4).

All of the solutions we have obtained up to now belong to the linearizable class which exist for integer and half-integer values of c. Whenever the independent variable z also takes integer values (this means that the offset z_0 of the origin must be an integer) the solution either involves the error-function or even becomes rational. However, these rational solutions are not the only ones for P_{IV}/asymmetric d-P_{I}. Another family of rational solutions does exist, outside the linearizable class. These rational solutions exist for integer z and c

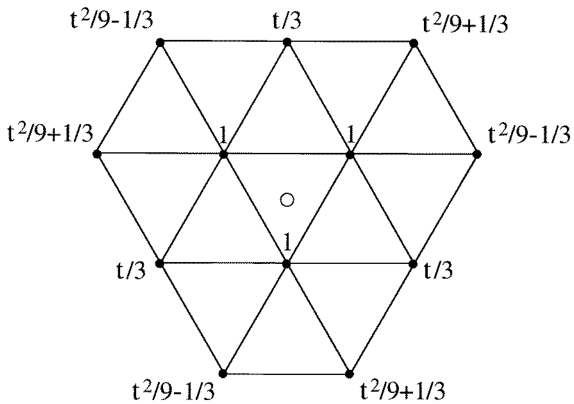

Figure 3. The simplest τ-functions for the construction of rational solutions. The ○ denotes the origin of the z, c coordinates.

of the form $c = m - \sigma/3$ with integer m, and also for half-integer z and $c = m + \sigma/6$, with $\sigma = 1$ *or* $\sigma = -1$. In figure 3 (drawn for $\sigma = 1$) we present the simplest τ-functions of these rational solutions. It is clear that through a global scaling and the two available gauges we can choose the three τ-functions nearest to the origin to have values $\tau = 1$. Next we have the three next-nearest neighbours which are all taken to be equal (which means that the solution is of codimension two), and depend in principle on the parameter t. Since the bilinear equation (3.9) does not involve t, we can take this common value arbitrarily. Returning from the τ's to the x, y, r we reconstruct (3.1) we obtain on the right-hand side instead of t the triple of the assumed common value of the τ's. Thus this value must be exactly $t/3$. Using these seed solutions we can construct the higher τ-functions by iterating (3.9). Because (3.9) is a Hirota–Miwa equation satisfying the singularity confinement criterion the obtained τ's turn out to be polynomials in t, i.e. the necessary factorizations do occur. Note that since both figure 3 and (3.9) are invariant under a $\pm 2\pi/3$ rotation, this will be true of the whole τ-plane. Thus one has $\tau(z, c) = \tau((-z \pm 3c)/2, (-c \mp z)/2)$ (recall that the orthonormal values of the coordinates are $2z/\sqrt{3}$ and $2c$ respectively). On the other hand (3.9) does not have a reflection symmetry and thus in general $\tau(-z, c) \neq \tau(z, c)$ though a relation exists between these quantities namely,

$$\tau(-z, c; t) = (-\mathrm{i})^N \tau(z, c; \mathrm{i}t) \tag{4.6}$$

where N is their common degree. We obtain the following τ's:

$$\tau(\tfrac{1}{2}, \tfrac{1}{6}) = \tau(-\tfrac{1}{2}, \tfrac{1}{6}) = \tau(0, -\tfrac{1}{3}) = 1 \tag{4.7a}$$

$$\tau(0, \tfrac{2}{3}) = \tau(1, -\tfrac{1}{3}) = \tau(-1, -\tfrac{1}{3}) = t/3 \tag{4.7b}$$

$$\tau(1, \tfrac{2}{3}) = \tau(\tfrac{1}{2}, -\tfrac{5}{6}) = \tau(-\tfrac{3}{2}, \tfrac{1}{6}) = t^2/9 + \tfrac{1}{3} \tag{4.7c}$$

$$\tau(-1, \tfrac{2}{3}) = \tau(-\tfrac{1}{2}, -\tfrac{5}{6}) = \tau(\tfrac{3}{2}, \tfrac{1}{6}) = t^2/9 - \tfrac{1}{3}. \tag{4.7d}$$

From now on we will use the shorthand $T = t^2/3$:

$$\tau(\pm\tfrac{1}{2}, \tfrac{7}{6}) = \tau(\pm\tfrac{3}{2}, -\tfrac{5}{6}) = \tau(\mp 2, -\tfrac{1}{3}) = (T^2 \pm 2T - 1)/9 \tag{4.7e}$$

$$\tau(2, \tfrac{2}{3}) = \tau(-2, \tfrac{2}{3}) = \tau(0, -\tfrac{4}{3}) = t(T^2 - 5)/27 \tag{4.7f}$$

$$\tau(\pm\tfrac{3}{2}, \tfrac{7}{6}) = \tau(\pm 1, -\tfrac{4}{3}) = \tau(\mp\tfrac{5}{2}, \tfrac{1}{6}) = (T^3 \pm 5T^2 + 5T \pm 5)/27 \tag{4.7g}$$

$$\tau(0, \tfrac{5}{3}) = \tau(\tfrac{5}{2}, -\tfrac{5}{6}) = \tau(-\tfrac{5}{2}, -\tfrac{5}{6}) = (T^4 - 14T^2 - 7)/81 \tag{4.7h}$$

$$\tau(\pm 1, \tfrac{5}{3}) = \tau(\pm 2, -\tfrac{4}{3}) = \tau(\mp 3, -\tfrac{1}{3}) = t(T^4 \pm 8T^3 + 14T^2 - 35)/243 \tag{4.7i}$$

$$\begin{aligned}\tau(\pm\tfrac{5}{2}, \tfrac{7}{6}) &= \tau(\pm\tfrac{1}{2}, -1\tfrac{1}{6}) = \tau(\mp 3, \tfrac{2}{3})\\ &= (T^5 \pm 5T^4 - 10T^3 \mp 50T^2 - 75T \pm 25)/243\end{aligned} \tag{4.7j}$$

$$\begin{aligned}\tau(\pm 2, \tfrac{5}{3}) &= \tau(\pm\tfrac{3}{2}, -1\tfrac{1}{6}) = \tau(\mp\tfrac{7}{2}, \tfrac{1}{6})\\ &= (T^6 \pm 14T^5 + 65T^4 \pm 140T^3 + 175T^2 \pm 350T + 175)/3^6\end{aligned} \tag{4.7k}$$

$$\begin{aligned}\tau(\pm\tfrac{1}{2}, \tfrac{13}{6}) &= \tau(\pm 3, -\tfrac{4}{3}) = \tau(\mp\tfrac{7}{2}, -\tfrac{5}{6})\\ &= (T^7 \pm 7T^6 - 21T^5 \mp 175T^4 - 245T^3 \pm 245T^2 - 735T \mp 245)/3^7\end{aligned} \tag{4.7l}$$

$$\tau(\tfrac{7}{2}, \tfrac{7}{6}) = \tau(-\tfrac{7}{2}, \tfrac{7}{6}) = \tau(0, -\tfrac{7}{3}) = (T^8 - 60T^6 + 550T^4 - 5500T^2 - 1375)/3^8 \tag{4.7m}$$

$$\begin{aligned}\tau(\pm\tfrac{3}{2}, \tfrac{13}{6}) &= \tau(\pm\tfrac{5}{2}, -1\tfrac{1}{6}) = \tau(\mp 4, -\tfrac{1}{3})\\ &= (T^8 \pm 20T^7 + 140T^6 \pm 420T^5 + 350T^4 \mp 980T^3\\ &\quad -4900T^2 \mp 4900T + 1225)/3^8.\end{aligned} \tag{4.7n}$$

Note in that (4.7f), (4.7h) and (4.7m) one has $\tau(3c, c) = \tau(-3c, c)$, but this is related to the fact that these points correspond to each other, not only by a reflection, but also by a rotation by $2\pi/3$, as for (4.7a, b).

In contrast to the case of the linearizable solutions, we cannot give a Casorati form for the higher τ's.

One interesting thing is that the degree of the τ as a polynomial in t can be explicitly constructed. We find that for a point with coordinates (z, c) we have $N = (9d^2 - 4)/12$, where d is the distance to the origin: $d^2 = 4z^2/3 + 4c^2$ (since the orthonormal values of the coordinates are $2z/\sqrt{3}$ and $2c$ respectively). This degree, however, does not uniquely characterize the polynomial τ. We have already $\tau(1, \frac{2}{3}) \neq \tau(-1, \frac{2}{3})$, though these two τ's are related by (4.6), as we said earlier. However, the situation is even more complicated since we can find polynomials with the same degree but without any relation. This occurs for the first time at $N = 16$ as can be seen in (4.7m, n).

5. Conclusion

This paper deals with the derivation of the asymmetric d-$\mathrm{P_I}$ equation starting from the continuous $\mathrm{P_{IV}}$. This relation is not limited to these particular equations. As a matter of fact, all of the discrete, *difference* Painlevé equations can be obtained from the Schlesinger's of the continuous ones. This leads quite naturally to the Lax pair of the d-P's. Moreover this procedure produces the most general form of the d-$\mathbb{P}$ here the 'asymmetric' d-$\mathrm{P_I}$: no degree of freedom is lost. This construction explains the property of self-duality and provides the basis of the classification of the d-$\mathbb{P}$'s. However one must keep in mind that for the full classification of *all* the d-$\mathbb{P}$'s one may have to consider higher (continuous) Garnier problems. Moreover there exists a class of discrete equations, the (multiplicative) q-$\mathbb{P}$'s which are outside this approach. One interesting result of the self-dual approach is that the Hirota–Miwa equation is omnipresent. This is in perfect parallel to the appearance of the Toda equation in Okamoto's description of τ-sequences [18]. Finally the parallel description of the continuous and discrete Painlevé equations we presented here allows a simultaneous construction of the special solutions of both. It should be interesting to extend this approach to the remaining discrete Painlevé equations.

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